

# On Gauss-Pólya's Inequality

By

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## Abstract

Let  $g, b: [a, b] \rightarrow \mathbf{R}$  be nonnegative nondecreasing functions such that  $g$  and  $b$  have a continuous first derivative and  $g(a) = b(a), g(b) = b(b)$ . Let  $p = (p_1, p_2)$  be a pair of positive real numbers  $p_1, p_2$  such that  $p_1 + p_2 = 1$ .

a) If  $f: [a, b] \rightarrow \mathbf{R}$  be a nonnegative nondecreasing function, then for  $r, s < 1$

$$M_p^{[r]} \left( \int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \leq \int_a^b (M_p^{[s]}(g(t), b(t)))' f(t) dt \quad (1)$$

holds, and for  $r, s > 1$  the inequality is reversed.

b) If  $f: [a, b] \rightarrow \mathbf{R}$  is a nonnegative nonincreasing function then for  $r < 1 < s$  (1) holds and for  $r > 1 > s$  the inequality is reversed.

Similar results are derived for quasiarithmetic and logarithmic means.

*Key words and phrases:* Logarithmic mean, quasiarithmetic mean, Pólya's inequality, weighted mean.

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## 1. Introduction

Gauss mentioned the following result in [2]:

*If  $f$  is a nonnegative and decreasing function then*

$$\left( \int_0^\infty x^2 f(x) dx \right)^2 \leq \frac{5}{9} \int_0^\infty f(x) dx \int_0^\infty x^4 f(x) dx. \quad (2)$$

Pólya and Szegő classical book “Problems and Theorems in Analysis, I” [7] gives the following generalization and extension of Gauss’ inequality (2).

**Theorem A.** (Pólya’s inequality) *Let  $a$  and  $b$  be nonnegative real numbers. a) If  $f: [0, \infty) \rightarrow \mathbf{R}$  is a nonnegative and decreasing function, then*

$$\left( \int_0^\infty x^{a+b} f(x) dx \right)^2 \leq \left( 1 - \left( \frac{a-b}{a+b+1} \right)^2 \right) \int_0^\infty x^{2a} f(x) dx \\ \times \int_0^\infty x^{2b} f(x) dx \quad (3)$$

*whenever the integrals exist.*

b) If  $f: [0, 1) \rightarrow \mathbf{R}$  is a nonnegative and increasing function, then

$$\left( \int_0^1 x^{a+b} f(x) dx \right)^2 \geq \left( 1 - \left( \frac{a-b}{a+b+1} \right)^2 \right) \int_0^1 x^{2a} f(x) dx \\ \times \int_0^1 x^{2b} f(x) dx. \quad (4)$$

Obviously, putting  $a = 0$  and  $b = 2$  in (3) we obtain Gauss’ inequality. Recently Pečarić and Varošanec [6] obtained a generalization.

**Theorem B.** *Let  $f: [a, b] \rightarrow \mathbf{R}$  be nonnegative and increasing, and let  $x_i: [a, b] \rightarrow \mathbf{R}$  ( $i = 1, \dots, n$ ) be nonnegative increasing functions with a continuous first derivative. If  $p_i$ , ( $i = 1, \dots, n$ ) are positive real numbers such that  $\sum_{i=1}^n \frac{1}{p_i} = 1$ , then*

$$\int_a^b \left( \prod_{i=1}^n (x_i(t))^{1/p_i} \right)' f(t) dt \geq \prod_{i=1}^n \left( \int_a^b x_i'(t) f(t) dt \right)^{1/p_i} \quad (5)$$

*If  $x_i(a) = 0$  for all  $i = 1, \dots, n$  and if  $f$  is a decreasing function then the reverse inequality holds.*

The previous result is an extension of the Pólya’s inequality. If we substitute in (5):  $n = 2, p_1 = p_2 = 2, a = 0, b = 1, g(x) = x^{2u+1}, h(x) = x^{2v+1}$  where  $u, v > 0$ , we have (4).

In this paper we provide generalizations of Theorem B in a number of directions. In Section 2 we first provide the inequality for weighted means. We note that, as is suggested by notation for means, our result extends to the case when the ordered pair  $(p_1, p_2)$  is replaced by an  $n$ -tuple. We derive also a version of our theorem for higher derivatives.

Section 4 treats some corresponding results when  $M$  is replaced by quasiarithmetic mean. This can be done when the function involved enjoys appropriate convexity properties. A second theorem in Section 4 allows one weight  $p_1$  to be positive and the others negative.

Section 5 addresses the logarithmic mean.

## 2. Results Connected with Weighted Means

$M_p^{[r]}(a)$  denotes the weighted mean of order  $r$  and weights  $p = (p_1, \dots, p_n)$  of a positive sequence  $a = (a_1, \dots, a_n)$ . The  $n$ -tuple  $p$  is of positive numbers  $p_i$  with  $\sum_{i=1}^n p_i = 1$ . The mean is defined by

$$M_p^{[r]}(a) = \begin{cases} \left( \sum_{i=1}^n p_i a_i^r \right)^{1/r} & \text{for } r \neq 0 \\ \prod_{i=1}^n a_i^{p_i} & \text{for } r = 0. \end{cases}$$

In the special cases  $r = -1, 0, 1$  we obtain respectively the familiar harmonic, geometric and arithmetic mean.

The following theorem, which is a simple consequence of Jensen's inequality for convex functions, is one of the most important inequalities between means.

**Theorem C.** *If  $a$  and  $p$  are positive  $n$ -tuples and  $s < t$ ,  $s, t \in \mathbf{R}$ , then*

$$M_p^{[s]}(a) \leq M_p^{[t]}(a) \quad \text{for } s < t, \quad (6)$$

*with equality if and only if  $a_1 = \dots = a_n$ .*

A well-known consequence of the above statement is the inequality between arithmetic and geometric means. Previous results and refinements can be found in [3].

The following theorem is the generalization of Theorem B.

**Theorem 1.** *Let  $g, h: [a, b] \rightarrow \mathbf{R}$  be nonnegative nondecreasing functions such that  $g$  and  $h$  have a continuous first derivative and  $g(a) = h(a), g(b) = h(b)$ . Let  $p = (p_1, p_2)$  be a pair of positive real numbers  $p_1, p_2$  such that  $p_1 + p_2 = 1$ .*

a) *If  $f: [a, b] \rightarrow \mathbf{R}$  be a nonnegative nondecreasing function, then for  $r, s < 1$*

$$M_p^{[r]} \left( \int_a^b g'(t) f(t) dt, \int_a^b h'(t) f(t) dt \right) \leq \int_a^b \left( M_p^{[s]}(g(t), h(t)) \right)' f(t) dt \quad (7)$$

*holds, and for  $r, s > 1$  the inequality is reversed.*

b) If  $f: [a, b] \rightarrow \mathbf{R}$  is a nonnegative nonincreasing function then for  $r < 1 < s$  (7) holds and for  $r > 1 > s$  the inequality is reversed.

*Proof:* Let us suppose that  $r, s < 1$  and  $f$  is nondecreasing. Using inequality (6) we obtain

$$\begin{aligned}
 & M_p^{[r]} \left( \int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \\
 & \leq M_p^{[1]} \left( \int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \\
 & = \int_a^b (p_1 g'(t) + p_2 b'(t)) f(t) dt \\
 & = f(b) M_p^{[1]}(g(b), b(b)) - f(a) M_p^{[1]}(g(a), b(a)) \\
 & \quad - \int_a^b M_p^{[1]}(g(t), b(t)) df(t) \\
 & \leq f(b) M_p^{[1]}(g(b), b(b)) - f(a) M_p^{[1]}(g(a), b(a)) \\
 & \quad - \int_a^b M_p^{[s]}(g(t), b(t)) df(t) \\
 & = f(b) M_p^{[1]}(g(b), b(b)) - f(a) M_p^{[1]}(g(a), b(a)) \\
 & \quad - \left( f(b) M_p^{[s]}(g(b), b(b)) - f(a) M_p^{[s]}(g(a), b(a)) \right. \\
 & \quad \left. - \int_a^b (M_p^{[s]}(g(t), b(t)))' f(t) dt \right) \\
 & = f(b) \left( M_p^{[1]}(g(b), b(b)) - M_p^{[s]}(g(b), b(b)) \right) \\
 & \quad - f(a) \left( M_p^{[1]}(g(a), b(a)) - M_p^{[s]}(g(a), b(a)) \right) \\
 & \quad + \int_a^b \left( M_p^{[s]}(g(t), b(t)) \right)' f(t) dt \\
 & = \int_a^b \left( M_p^{[s]}(g(t), b(t)) \right)' f(t) dt.
 \end{aligned}$$

A similar proof applies in each of the other cases.  $\square$

**Remark 1.** In Theorem 1 we deal with two functions  $g$  and  $b$ . Obviously a similar result holds for  $n$  functions  $x_1, \dots, x_n$  which satisfy the same conditions as  $g$  and  $b$ .

**Remark 2.** It is obvious that on substituting  $r = s = 0$  into (7) we have inequality (5) for  $n = 2$ . The result for  $r = s = 0$  is given in [1].

In the following theorem we consider an inequality involving higher derivatives.

**Theorem 2.** Let  $f: [a, b] \rightarrow \mathbf{R}$ ,  $x_i: [a, b] \rightarrow \mathbf{R}$  ( $i = 1, \dots, m$ ) be nonnegative functions with continuous  $n$ -th derivatives such that  $x_i^{(n)}$ , ( $i = 1, \dots, m$ ) are nonnegative functions and  $p_i$ , ( $i = 1, \dots, m$ ) be positive real numbers such that  $\sum_{i=1}^m p_i = 1$ .

a) If  $(-1)^{n-1} f^{(n)}$  is a nonnegative function, then for  $r, s < 1$

$$\begin{aligned} & M_p^{[r]} \left( \int_a^b x_1^{(n)}(t) f(t) dt, \dots, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\ & \leq \Delta + \int_a^b \left( M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(n)} f(t) dt \end{aligned} \quad (8)$$

holds, where

$$\begin{aligned} \Delta = & \sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \\ & \left. \left( \sum_{i=1}^m p_i x_i^{(k)}(t) - \left( M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(k)} \right) \right|_a^b \end{aligned}$$

If

$$x_i^{(k)}(a) = x_j^{(k)}(a) \text{ and } x_i^{(k)}(b) = x_j^{(k)}(b) \text{ for } i, j \in \{1, \dots, m\} \quad (9)$$

and  $k = 0, \dots, n-1$ , then

$$\begin{aligned} & M_p^{[r]} \left( \int_a^b x_1^{(n)}(t) f(t) dt, \dots, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\ & \leq \int_a^b \left( M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(n)} f(t) dt. \end{aligned} \quad (10)$$

If  $r, s > 1$ , then the inequalities (8) and (10) are reversed.

b) If  $(-1)^n f^{(n)}$  is a nonnegative function, then for  $r < 1 < s$  the inequalities (8) and (10) hold and for  $r > 1 > s$  they are reversed.

*Proof:* a) Let  $r$  and  $s$  be less than 1. Integrating by part  $n$ -times and using (6), we obtain

$$\begin{aligned}
 & M_p^{[r]} \left( \int_a^b x_1^{(n)}(t) f(t) dt, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\
 & \leq M_p^{[1]} \left( \int_a^b x_1^{(n)}(t) f(t) dt, \dots, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\
 & = \left( \sum_{k=0}^{n-1} (-1)^{n-k} f^{(n-k-1)}(t) \sum_{i=1}^m p_i x_i^{(k)}(t) \right) \Big|_a^b \\
 & \quad - \int_a^b M_p^{[1]}(x_1(t), \dots, x_m(t)) (-1)^{n-1} f^{(n)}(t) dt \\
 & \leq \left( \sum_{k=0}^{n-1} (-1)^{n-k} f^{(n-k-1)}(t) \sum_{i=1}^m p_i x_i^{(k)}(t) \right) \Big|_a^b \\
 & \quad - \int_a^b M_p^{[s]}(x_1(t), \dots, x_m(t)) (-1)^{n-1} f^{(n)}(t) dt \\
 & = \Delta + \int_a^b \left( M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(n)} f(t) dt.
 \end{aligned}$$

We shall prove that  $\Delta = 0$  if  $x_i, i = 1, \dots, m$ , satisfy (9).

Let us use notation  $A_k = x_i^{(k)}(a)$  for  $k = 0, 1, \dots, n-1$ . Then  $\sum_{i=1}^m p_i x_i^{(k)}(a) = A_k$ . Consider the  $k$ -th order derivative of function  $y^p$  where  $y$  is an arbitrary function with  $k$ -th order derivative. First, there exists function  $\phi_k^{[p]}$  such that

$$(y^p)^{(k)} = \phi_k^{[p]}(y, y', \dots, y^{(k)}).$$

This follows by induction on  $k$ . For  $k = 1$  we have  $(y^p)' = py^{p-1}y' = \phi_1^{[p]}(y, y')$ . Suppose that proposition is valid for all  $j < k+1$ . Then using Leibniz's rule we get

$$\begin{aligned}
 (y^p)^{(k+1)} &= (py^{p-1}y')^{(k)} \\
 &= p \sum_{j=0}^k \binom{k}{j} (y^{p-1})^{(j)} (y')^{(k-j)} \\
 &= p \sum_{j=0}^k \binom{k}{j} \phi_j^{[p-1]}(y, y', \dots, y^{(j)}) y^{(k-j+1)} \\
 &= \phi_{k+1}^{[p]}(y, y', \dots, y^{(k+1)}).
 \end{aligned} \tag{11}$$

Suppose that  $s \neq 0$  and use the abbreviated notation  $M(t)$  for the mean  $M_p^{[s]}(x_1(t), \dots, x_m(t))$ . Then  $M^s(t) = \sum_{i=1}^m P_i x_i^s(t)$ . The statement " $M^{(k)}(a) = A_k$ " will be proved by induction on  $k$ . It is easy to check for  $k = 0$  and  $k = 1$ .

Suppose it holds for all  $j < k + 1$ . Then

$$\begin{aligned} \left( \sum_{i=1}^m p_i x_i^s(t) \right)^{(k+1)} \Big|_{t=a} &= \sum_{i=1}^m p_i \phi_{(k+1)}^{[s]}(x_i(t), x'_i(t), \dots, x_i^{(k+1)}(t)) \Big|_{t=a} \\ &= \phi_{(k+1)}^{[s]}(A_0, A_1, \dots, A_{k+1}) \\ &= s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(A_0, A_1, \dots, A_j) A_{k-j+1} \\ &\quad + \phi_k^{[s-1]}(A_0, A_1, \dots, A_k) A_{k+1}. \end{aligned}$$

On the other hand, using (11) we get

$$\begin{aligned} (M^s(t))^{(k+1)} \Big|_{t=a} &= s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(M(a), M'(a), \dots, M^{(j)}(a)) \\ &\quad \times M^{(k-j+1)}(a) + \phi_k^{[s-1]}(M(a), M'(a), \dots, M^{(k)}(a)) M^{(k+1)}(a) \\ &= s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(A_0, A_1, \dots, A_j) A_{k-j+1} + \phi_k^{[s-1]} \\ &\quad (A_0, A_1, \dots, A_k) M^{(k+1)}(a). \end{aligned}$$

Comparing these two results we obtain that  $M^{(k+1)}(a) = A_{k+1}$ , which is enough to conclude that  $\Delta = 0$ .

In the other cases the proof is similar, except in the case  $s = 0$  which is left to the reader.  $\square$

### 3. Applications

Now we will restrict our attention to the case when  $r = 0$  and the  $x_i$  are power functions.

**The case when  $n = 1$ .**

Set:  $r = 0, n = 1, a = 0, b = 1, x_i(t) = t^{a_i p_i + 1}$  in (8), where  $a_i > -\frac{1}{p_i}$  for  $i = 1, \dots, m, p_i > 0$  and  $\sum_{i=1}^m \frac{1}{p_i} = 1$ . We obtain that  $\Delta = 0$  and

$$\int_0^1 t^{a_1 + \dots + a_m} f(t) dt \geq \frac{\prod_{i=1}^m (a_i p_i + 1)^{1/p_i}}{1 + \sum_{i=1}^m a_i} \prod_{i=1}^m \left( \int_0^1 t^{a_i p_i} f(t) dt \right)^{1/p_i} \quad (12)$$

if  $f$  is a nondecreasing function. It is an improvement of Polya's inequality (4). Some other results related to this inequality can be found in [5] and [8].

For example, combining (12) and the inequality

$$\sum_{i=1}^m a_i + 2 \geq \prod_{i=1}^m (a_i p_i + 2)^{1/p_i}$$

which follows from the inequality between arithmetic and geometric means, we obtain

$$\begin{aligned} \int_0^1 t^{a_1 + \dots + a_m} f(t) dt &\geq \frac{\prod_{i=1}^m ((a_i p_i + 1)(a_i p_i + 2))^{1/p_i}}{(1 + \sum_{i=1}^m a_i)(2 + \sum_{i=1}^m a_i)} \\ &\times \prod_{i=1}^m \left( \int_0^1 t^{a_i p_i} f(t) dt \right)^{1/p_i} \end{aligned} \quad (13)$$

**The case when  $n = 2$ .**

Set:  $r = 0, n = 2, a = 0, b = 1, x_i(t) = t^{a_i p_i + 2}$  in (8), where  $a_i > -\frac{1}{p_i}$  for  $i = 1, \dots, m, p_i > 0$  and  $\sum_{i=1}^m \frac{1}{p_i} = 1$ . After some simple calculation, we obtain that  $\Delta = 0$  and inequality (13) holds if  $f$  is a concave function. So inequality (13) applies not only for  $f$  nondecreasing, but also for  $f$  concave.

#### 4. Results for Quasiarithmetic Means

**Definition 2.** Let  $f$  be a monotone real function with inverse  $f^{-1}, p = (p_1, \dots, p_n) = (p_i)_i, a = (a_1, \dots, a_n) = (a_i)_i$  be real  $n$ -tuples. The quasiarithmetic mean of  $n$ -tuple  $a$  is defined by

$$M_f(a; p) = f^{-1} \left( \frac{1}{P_n} \sum_{i=1}^n p_i f(a_i) \right),$$

where  $P_n = \sum_{i=1}^n p_i$ .

For  $p_i \geq 0, P_n = 1, f(x) = x^r (r \neq 0)$  and  $f(x) = \ln x (r = 0)$  the quasiarithmetic mean  $M_f(a; p)$  is the weighted mean  $M_p^{[r]}(a)$  of order  $r$ .

**Theorem 3.** Let  $p$  be a positive  $n$ -tuple,  $x_i: [a, b] \rightarrow \mathbf{R} (i = 1, \dots, n)$  be non-negative functions with continuous first derivative such that  $x_i(a) = x_j(a), x_i(b) = x_j(b), i, j = 1, \dots, n$

a) If  $\varphi$  is a nonnegative nondecreasing function on  $[a, b]$  and if  $f$  and  $g$  are convex increasing or concave decreasing functions, then

$$M_f \left( \left( \int_a^b x_i'(t) \varphi(t) dt \right)_i; p \right) \geq \int_a^b M_g'((x_i(t))_i; p) \varphi(t) dt. \quad (14)$$

If  $f$  and  $g$  are concave increasing or convex decreasing functions, the inequality is reversed.

b) If  $\varphi$  is a nonnegative nonincreasing function on  $[a, b]$ ,  $f$  convex increasing or concave decreasing function and  $g$  is concave increasing or convex decreasing, then (14) holds.

If  $f$  is concave increasing or convex decreasing function and  $g$  is convex increasing or concave decreasing, then (14) is reversed.

*Proof:* Suppose that  $\varphi$  is nondecreasing and  $f$  and  $g$  are convex functions. We shall use integration by parts and the well-known Jensen inequality for convex functions. The latter states that if  $(p_i)$  is a positive  $n$ -tuple and  $a_i \in I$ , then for every convex function  $f: I \rightarrow \mathbb{R}$  we have

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i a_i\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(a_i). \quad (15)$$

We have

$$\begin{aligned} M_f\left(\left(\int_a^b x'_i(t) \varphi(t) dt\right)_i; p\right) &= f^{-1}\left(\frac{1}{P_n} \sum_{i=1}^n p_i f\left(\int_a^b x_i(t) \varphi(t) dt\right)\right) \\ &\geq \frac{1}{P_n} \sum_{i=1}^n p_i \int_a^b x'_i(t) \varphi(t) dt = \int_a^b \frac{1}{P_n} \left(\sum_{i=1}^n p_i x'_i(t)\right) \varphi(t) dt \\ &= \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t) \varphi(t) \Big|_a^b - \int_a^b \frac{1}{P_n} \left(\sum_{i=1}^n p_i x_i(t)\right) d\varphi(t) \\ &\geq \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t) \varphi(t) \Big|_a^b - \int_a^b g^{-1}\left(\frac{1}{P_n} \left(\sum_{i=1}^n p_i g(x_i(t))\right)\right) d\varphi(t) \\ &= \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t) \varphi(t) \Big|_a^b - \int_a^b M_g(x_i(t))_i; p d\varphi(t) \\ &= \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t) \varphi(t) \Big|_a^b - M_g((x_i(t))_i; p) \varphi(t) \Big|_a^b \\ &\quad + \int_a^b M'_g((x_i(t))_i; p) \varphi(t) dt = \int_a^b M'_g((x_i(t))_i; p) \varphi(t) dt. \quad \square \end{aligned}$$

**Theorem 4.** Let  $x_i, i = 1, \dots, n$ , satisfy assumptions of Theorem 4 and let  $p$  be a real  $n$ -tuple such that

$$p_1 > 0, \quad p_i \leq 0 \quad (i = 2, \dots, n), \quad P_n > 0. \quad (16)$$

a) If  $\varphi$  is a nonnegative nonincreasing function on  $[a, b]$  and if  $f$  and  $g$  are concave increasing or convex decreasing functions, then (14) holds, while if  $f$  and  $g$  are convex increasing or concave decreasing (14) is reversed.

b) If  $\varphi$  is a nonnegative nondecreasing function on  $[a, b]$ ,  $f$  is convex increasing or concave decreasing and  $g$  concave increasing or convex decreasing, then (14) holds.

If  $f$  is concave increasing or convex decreasing and  $g$  is convex increasing or concave decreasing, then (14) is reversed.

The proof is similar to that of Theorem 4. Instead of Jensen's inequality, a reverse Jensen's inequality [3, p. 6] is used: that is, if  $p$  is real  $n$ -tuple such that (16) holds,  $a_i \in I$ ,  $i = 1, \dots, n$ , and  $(1/P_n) \sum_{i=1}^n p_i a_i \in I$ , then for every convex function  $f: I \rightarrow \mathbb{R}$  (15) is reversed.

**Remark 3.** In Theorem 4 and 5 we deal with first derivatives. We can state an analogous result for higher-order derivatives as in Section 2.

**Remark 4.** The assumption that  $p$  is a positive  $n$ -tuple in Theorem 4 can be weakened to  $p$  being a real  $n$ -tuple such that

$$0 \leq \sum_{i=1}^k p_i \leq P_n \quad (1 \leq k \leq n), \quad P_n > 0$$

and  $(\int x'_i(t) \varphi(t) dt)_i$ , and  $(x_i(t))_i$ ,  $t \in [a, b]$  being monotone  $n$ -tuples.

In that case, we use Jensen-Steffensen's inequality [3, p. 6], instead of Jensen's in-equality in the proof.

In Theorem 5, the assumption on  $n$ -tuple  $p$  can be replaced by  $p$  being a real  $n$ -tuple such that for some  $k \in \{1, \dots, m\}$

$$\sum_{i=1}^k p_i \leq 0 (k < m) \quad \text{and} \quad \sum_{i=k}^n p_i \leq 0 (k > m)$$

and  $(\int x'_i(t) \varphi(t) dt)_i$ ,  $(x_i(t))_i$ ,  $t \in [a, b]$  being monotone  $n$ -tuples.

We use the reverse Jensen-Steffensen's inequality (see [3, p. 6] and [4]) in the proof.

## 5. Results for Logarithmic Means

We define the logarithmic mean  $L_r(x, y)$  of distinct positive numbers  $x, y$  by

$$L_r(x, y) = \begin{cases} \left( \frac{1}{y-x} \frac{y^{r+1} - x^{r+1}}{r+1} \right)^{1/r} & r \neq -1, 0 \\ \frac{1}{e} \left( \frac{y^y}{x^x} \right)^{\frac{1}{y-x}} & r = 0 \\ \frac{\ln y - \ln x}{y-x} & r = -1 \end{cases}$$

and take  $L_r(x, x) = x$ . The function  $r \mapsto L_r(x, y)$  is nondecreasing.

It is easy to see that  $L_1(x, y) = \frac{x+y}{2}$  and using method similar to that of the previous theorems we obtain the following result.

**Theorem 5.** Let  $g, h: [a, b] \rightarrow \mathbf{R}$  be nonnegative nondecreasing functions with continuous first derivatives and  $g(a) = h(a), g(b) = h(b)$ .

a) If  $f$  is a nonnegative increasing function on  $[a, b]$ , and if  $r, s \leq 1$ , then

$$L_r \left( \int_a^b g'(t)f(t) dt, \int_a^b h'(t)f(t) dt \right) \leq \int_a^b L'_s(g(t), h(t)f(t)) dt. \quad (16)$$

If  $r, s \geq 1$  then the reverse inequality holds.

b) If  $f$  is a nonnegative nonincreasing function then for  $r < 1 < s$  (16) holds, and for  $r > 1 > s$  the reverse inequality holds.

*Proof.* Let  $f$  be a nonincreasing function and  $r < 1 < s$ . Using  $F = -f$ , integration by parts and inequalities between logarithmic means we get

$$\begin{aligned} & L_r \left( \int_a^b g'(t)f(t) dt, \int_a^b h'(t)f(t) dt \right) \\ & \leq L_1 \left( \int_a^b g'(t)f(t) dt, \int_a^b h'(t)f(t) dt \right) = \frac{1}{2} \int_a^b (g(t) + h(t))' f(t) dt \\ & = \frac{1}{2} (g(t) + h(t))f(t) \Big|_a^b + \int_a^b \frac{1}{2} (g(t) + h(t)) dF(t) \\ & \leq \frac{1}{2} (g(t) + h(t))f(t) \Big|_a^b + \int_a^b L_s(g(t), h(t)) dF(t) \\ & = \frac{1}{2} (g(t) + h(t))f(t) \Big|_a^b - L_s(g(t), h(t))f(t) \Big|_a^b \\ & \quad + \int_a^b L'_s(g(t), h(t))f(t) dt = \int_a^b L'_s(g(t), h(t))f(t) dt. \end{aligned}$$

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